today:

homework 1 due (5.1.14, 5.2.6, 5.2.24, 5.3.29, 5.3.54, 5.4.26)
§ 5.6 - logarithms
§ 6.1 - area between curves

thursday:

review for midterm (come with questions)

friday:

mslc: webwork 2 workshop @ 11:30, 12:30, 1:30, 2:30, 3:30 in SE 040
webwork 2 due @ 11:55 pm

sunday:

mslc: midterm review 7:30 pm - 9:18 pm in HI 131

tuesday:

midterm: 5.1 - 5.6, 6.1
homework 2 due (5.5.36, 5.5.74, 5.6.4, 5.Review.38, 6.1.32, 6.1.48)

last time...

we introduced u substitution as a tool to help us find antiderivatives. It works by helping us recognize the result of a chain rule problem.

when done wisely, our u will be the inside function in the antiderivative.

last time...

we could also have simplified before differentiating by using trig relations.

tan(arcsin x)=x/sqrt(1-x^2)

sec(arcsin x)=1/sqrt(1-x^2)

Ask if there are u-substitution questions.

when done wisely, our u will be the inside function in the antiderivative.

example:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\tan\left(\arcsin x\right)\right) = \frac{\sec^2\left(\arcsin x\right)}{\sqrt{1-x^2}}$$

to integrate the right-hand side, we should choose $u = \arcsin\, x$, which happens to be the inside function of the composition on the left.

exploit symmetry

A function f is said to be even if $f(\text{-}x) = f(x)$

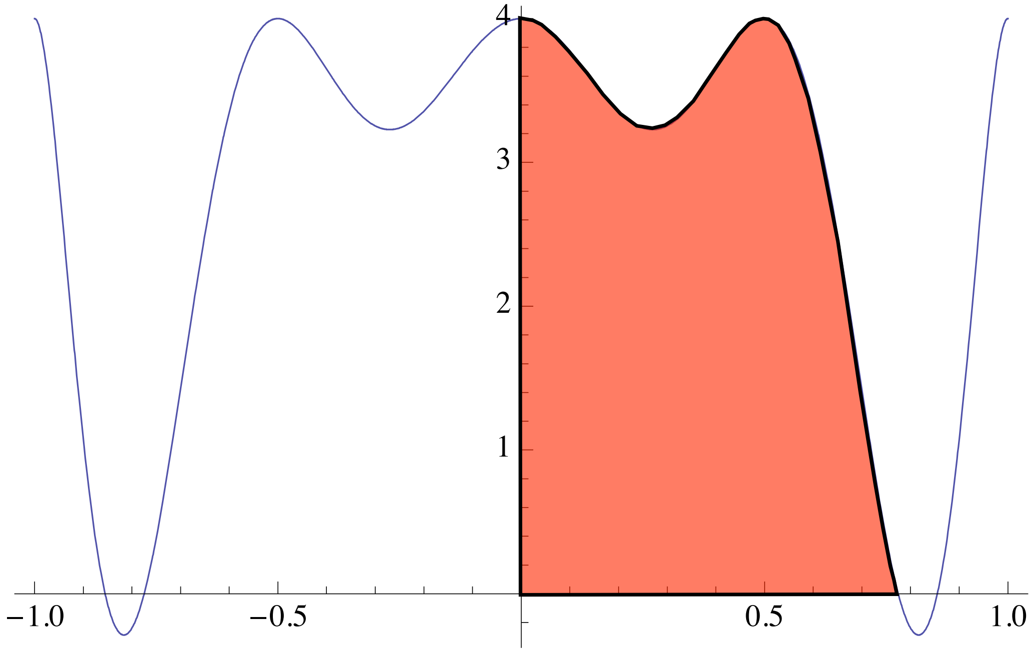
Examples of even functions include:

$$x^2$$
$$\cos(x)$$
$$\sin^2(x)$$
$$\sin\left(x^2\right)$$

even functions are symmetric across the y-axis.

$$f(x) = 4 + 100\left(x\left(x^2 - 1/4\right)\left(x^2 - 1\right)\right)^2\left(x^2 - 4\right)$$

exploit symmetry



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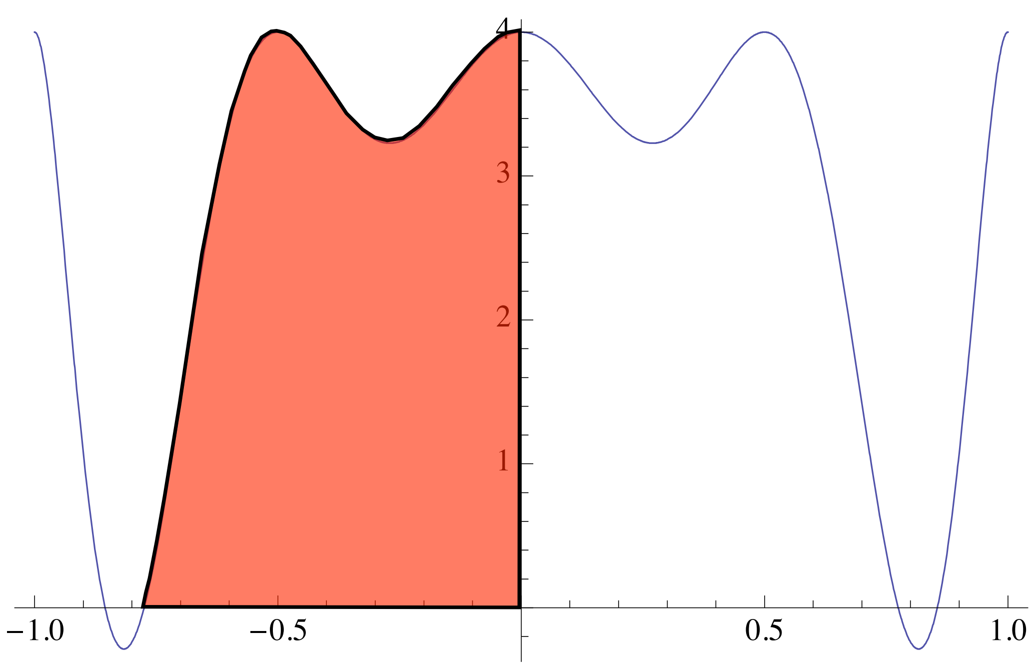
exploit symmetry



exploit symmetry

A function f is said to be odd if $f(-x) = -f(x)$

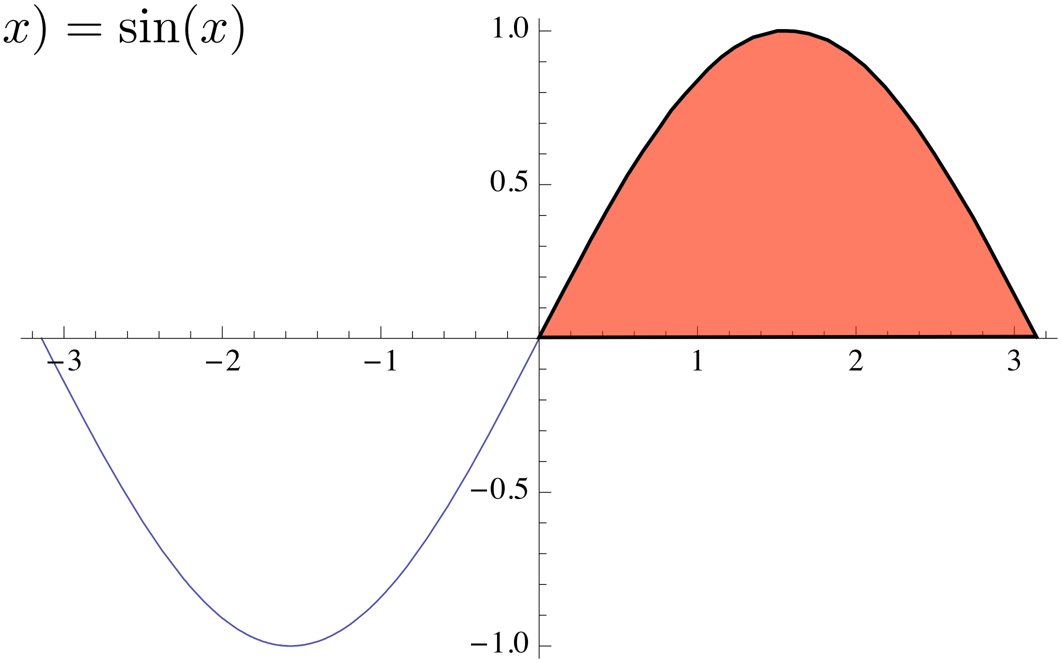
Examples of odd functions include:

$$x^3$$
$$\sin(x)$$
$$\tan(x)$$
$$x\ \cos(x)$$

odd functions are symmetric across the origin.

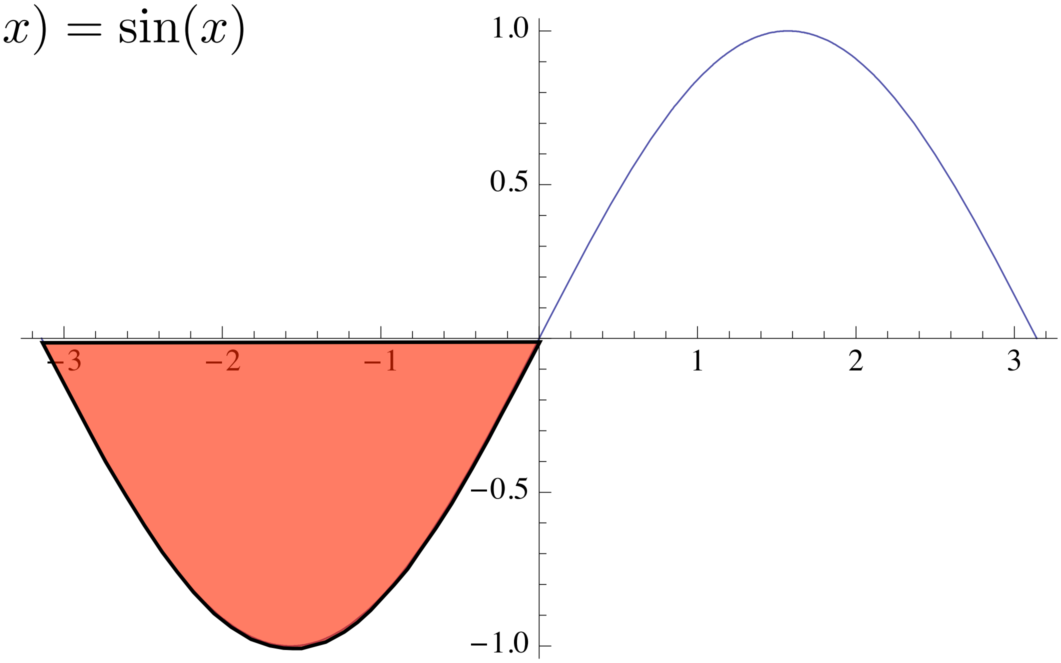
exploit symmetry

$f(x) = \sin(x)$



exploit symmetry

$f(x) = \sin(x)$



theorem

Note:

functions can be even, odd, or neither.

Let $a > 0$ be a real number
Let f be a function.

Then

If f is even, $$\int_{-a}^{a} f(x)\mathrm{d}x = 2\int_{0}^{a} f(x)\mathrm{d}x$$

If f is odd, $$\int_{-a}^{a} f(x)\mathrm{d}x = 0$$

exploit symmetry

with creativity, we can use the symmetry rules even if our function is neither even nor odd.

example:

$$\int_{-3}^{3} (x^2 + \sin x^3)\, \mathrm{d}x$$

integral = 18

what is $\ln x$?

usual definition is that ln is the inverse of exp, but that just shifts the question to what is e?

why is $\dfrac{\mathrm{d}}{\mathrm{d}x} e^x = e^x$?

Problem: there is no clue reason why this should be true if we just think of e as "some number"

In fact, why are the laws of exponents true?

logarithms

we define $\color{blue}{\ln x} := \displaystyle\int_1^x \frac{\mathrm{d}t}{t}$

Note that ln x is only defined for x>0. Why? Because the limit of the Reimann sum would not exist if we cross x=0, since 1/0 is undefined.

This is why there's an absolute value in the indefinite integral.

Points will be deducted if the antiderivative is written w/o the absolute value.

$$\int \frac{\mathrm{d}x}{x} = \ln|x| + C$$

logarithms

we define $\ln\, x := \int_1^x \frac{\mathrm{d}t}{t}$

all the normal properties of ln and exp follow from these definitions

For example, ln 1 = 0, since it represents no area

note that $1/x > 0$, so as x increases, $\ln\, x$ increases. Since $\ln\, x$ is a monotonic function, it is one-to-one, and hence has an inverse. We call this inverse $\exp\, x$.

theorem

let $x,\ y > 0$ be real numbers
let r be a rational number.

then:

$$\ln(xy) = \ln x + \ln y$$

$$\ln(x/y) = \ln x - \ln y$$

$$\ln\left(x^r\right) = r\, \ln x$$

The first two are proved in the book, so I will prove the third.

theorem

$\ln(x^r) = r \ln x$

proof:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln(x^r) - r \ln x\right) = \frac{r\,x^{r-1}}{x^r} - \frac{r}{x} = 0$$

thus $\ln(x^r) - r \ln x = c$ for some constant c.
plugging in $x{=}1$, we find $c{=}0$, proving the claim.

theorem

$$\frac{\mathrm{d}}{\mathrm{d}x}\exp x = \exp x$$

theorem

$$\frac{\mathrm{d}}{\mathrm{d}x}\exp x = \exp x$$

we know the derivative of ln y not because of some rule without proof from 151, but from the definition of ln and the first fundamental theorem of calculus

proof:

let $y = \exp x$.
then $\ln y = \ln(\exp x) = x$.
differentiating: $\dfrac{1}{y}\dfrac{\mathrm{d}y}{\mathrm{d}x} = 1$,
so $\dfrac{\mathrm{d}y}{\mathrm{d}x} = y = \exp x$.

theorem

That is, we define e to be the number such that the natural log of e is 1.

define $e := \exp(1)$. **then** $e^x = \exp(x)$.

theorem

define $e := \exp(1)$. then $e^x = \exp(x)$.

proof:

$$\ln(e^x) = x \ln e = x$$

A technical point: the theorem about ln putting exponents in front was only for rational numbers, not real numbers. Thus we define e^x := exp(x) when x is irrational.

the result follows immediately by taking $\exp$ (the inverse of $\ln$) of both sides.

theorem

Prove by letting f(x)=ln x. Then f'(x)=1/x, so f'(1)=1. Also compute derivative via definition, set equal, take exp of both sides.

Proof in book, p429, so omitted from notes.

$$e = \lim_{x \to 0} (1 + x)^{1/x}$$

theorem

let x, y be real numbers
let r be a rational number.

then:

$$e^{x+y} = e^x\, e^y$$

$$e^{x-y} = \frac{e^x}{e^y}$$

$$(e^x)^r = e^{r\,x}$$

As before, the last statement can be extended to apply to r real.

The book proves the first one; I prove the third.

theorem

$(e^x)^r = e^{r\,x}$

proof:

$$\ln\left((e^x)^r\right) = r\;\ln\left(e^x\right) = r\,x\;\ln e = r\,x$$

the result follows by taking $\exp$ of both sides.

theorem

let x be a real number, let $a,\ b > 0$.
we define $a^x := e^{x \ln a}$

then:

$$a^{x+y} = a^x\, a^y$$

$$a^{x-y} = a^x / a^y$$

$$(a^x)^y = a^{x\, y}$$

$$(a\, b)^x = a^x\, b^x$$

$$\frac{\mathrm{d}}{\mathrm{d}x} a^x = a^x\ \ln a$$

Prove these by invoking the definition of a^x and using the properties of e^x.

theorem

let x be a real number, let $a > 0$, $a \neq 1$.
we define $\log_a x$ to be the inverse of a^x.

then:

$$\log_a x = \frac{\ln x}{\ln a}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} (\log_a x) = \frac{1}{x\ \ln a}$$

The first follows since if y=log_a x, then a^y=x. Thus y ln a = ln x, so y=ln(x)/ln(a).

The second follows immediately from the first.

area between curves

example: Find the area of the region bounded by the line $x=0$ and the curves $y=\cos(x)$ and $y=\sin(x)$.

area between curves

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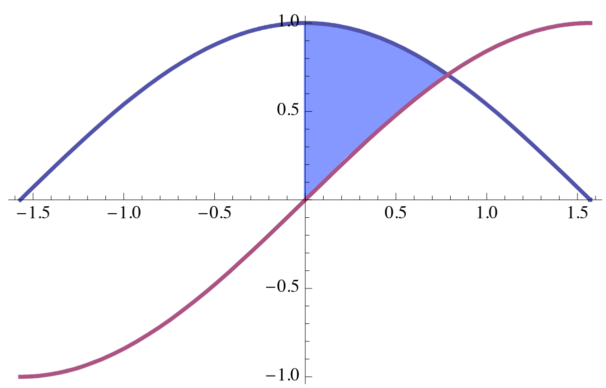
step one:
draw a picture

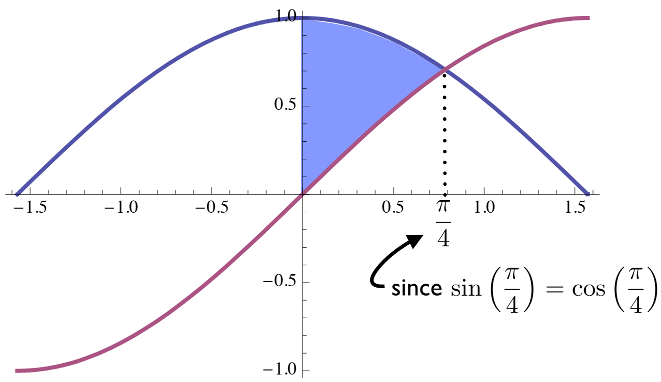
we are looking for the area of the blue shaded region



area between curves

example: Find the area of the region bounded by the line $x=0$ and the curves $y=\cos(x)$ and $y=\sin(x)$.

step two:
locate intersections by setting the functions equal to each other

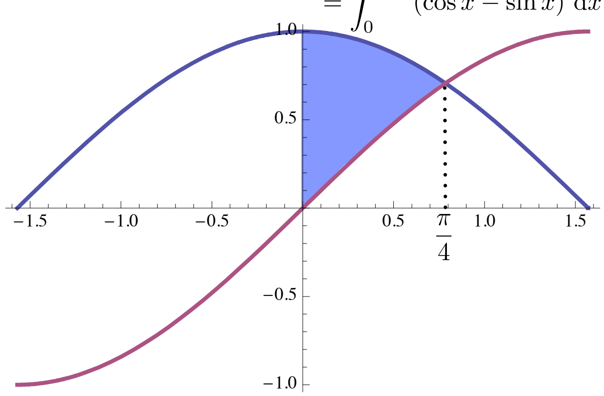


area between curves

example: Find the area of the region bounded by the line $x=0$ and the curves $y=\cos(x)$ and $y=\sin(x)$.

$$\text{area} = \int_0^{\pi/4} \cos x \, dx - \int_0^{\pi/4} \sin x \, dx$$
$$= \int_0^{\pi/4} (\cos x - \sin x) \, dx$$

step three:
area between the curves is area under top minus area under bottom curve



area between curves

suppose $f \geq g$ on $[a, b]$. Then the area between f and g on the interval is

$$\int_a^b (f(x) - g(x))\, \mathrm{d}x$$

We find area by integrating the height along the base. In 6.2, we'll find volume by integrating cross-sectional area along the 3rd dimension.

If sometimes $f \geq g$ and sometimes $g \geq f$, then you must find the areas of each piece separately and then combine.

It is sometimes useful to integrate with respect to y instead.

Problem 6.1.3 is a good example of when you would integrate with respect to y.

next time

- review 5.1 - 5.6, 6.1
- attempt "webwork midterm 1 review"
- come with questions
- webwork 2 due on friday

radical pi (undergrad math club) has a math/dungeons-and-dragons (?) themed talk on Wednesday at 5 in the undergrad math lounge (go down the stairs in the math building and it'll be on the left). Free pizza.